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ABSTRACT

This paper describes the BUD project which surveyed childrens' conceptions of division, and of fractions and decimals. The lack of connection between counting skills and conceptual understanding is discussed. The expectations for new algorithms and the basic idea in planning the BUD project are summarized. Some previous studies on counting, the development of everyday conceptions, and misconceptions of part of the whole concepts are reviewed. Principal research questions of this project are: (1) how children start dividing before they know the multiplication table and the division algorithm; (2) how children begin to understand the connection between division and fractions, and the meaning of words and written expressions; and (3) how children create a relationship between their conceptual knowledge of division and of rational numbers on the one hand and their arithmetic skills on the other. (YP)

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SCHOOL RESEARCH NEWSLETTER

JUNE 1989

FIELD-TOPIC

1989:4

Mathematics teaching

Current project:

CHILDREN'S CONCEPTIONS OF DIVISION AND OF
EQUAL PARTS. The BUD Project.

What is difficult about division?

And why do division and multiplication with fractions and
decimals give such a lot of trouble?

These questions, and various others connected with them, I
hope to be able to answer as a result of the inquiry to be
undertaken as part of the BUD project.

The work of the project takes as its starting point my own
previous research and also problems and questions, which,
time after time, have cropped up in research both in Sweden
and elsewhere.

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Quotitive and partitive division

One problem is how children conceive of division - whether they look on quotitive and partitive division as two different ways of counting or whether division is all one to them (1). In quotitive division one inquires after the number of times something can be extracted from the whole (e.g. "how many sixes in 48?"), while in partitive division one asks after the number of units in each part (e.g. "how many in each part if you divided 48 into 6 parts?").

A few decades ago, quotitive division was treated as one rule of arithmetic and partitive division as another in our textbooks. The "two rules of arithmetic" were even given separate signs ($:$ and $\div$ respectively).

Up till now, discussions of this question have centred round teachers' opinions concerning the way in which children conceive of division. Through the inquiry undertaken as part of the BUD project, on the other hand, I hope to be able to elucidate children's own conceptions of division.

Counting skills - conceptual knowledge

Another, more extensive problem studied in international surveys in recent years has been the lack of connection, observed in pupils' ways of solving problems (1 and 2), between counting skills and conceptual understanding.

Counting skills refer to knowledge of tables, proficiency in using algorithms, i.e. mathematical rules, etc. By conceptual knowledge, on the other hand, we mean children's understanding of what, say 7, 1,005 $\frac{3}{4}$, 0.4 or 5% and the "operational signs" +, $:$, =, etc. mean.

Intermediate-level pupils, for example, are often unable to understand why you have to multiply when working out the price of 0.6 kg cheese at SEK 68/kg but divide when working out the number of $\frac{3}{4}$ -hour lessons that can be got into six full hours. They sense that 0.6 kg cheese costs less than 1 kg and that $\frac{3}{4}$ -hours will outnumber complete hours. But multiplication, as they see it, results in "getting more" and division in "getting less". We seldom have time to give the children a chance of

relating these everyday conceptions to the counting skills we teach them. Mostly we have to make do, quite simply, with teaching them that you have to multiply in the first instance and divide in the second.

In Sweden too, we have now begun to discover and discuss this problem. We speak of "algorithmisation" in mathematics, meaning that too much time during mathematics lessons is spent drilling tables and rules. The pupils do not have the time or opportunity to develop their everyday conceptions into a more functional, conceptual knowledge. Nor do they have time to relate their everyday conceptions to the tables and algorithms which they learn in school.

To discuss this problem more fruitfully, perhaps we ought to start by pondering what we mean by "algorithm", which of course is the origin of the neologism "algorithmisation".

The word algorithm comes from the name of an Arab mathematician, al-Khowarazari, and above all it means "a rule for solving a mathematical problem".

I saw an interesting use made of the word in a television programme from the Open University in Britain, with reference, not to an addition or subtraction algorithm but to an algorithm for manufacturing a vehicle component. The speaker pointed out that the algorithm had to be designed first, and various means were then demonstrated of achieving this end: the finished component. Then came a no less important part in the construction of the algorithm, namely comparison of the different methods that had been used, or might conceivably be used, to achieve the end. The algorithm constructed was based on the least energy-demanding method.

Since the workers engaged on this particular task did not have to concern themselves with algorithms for manufacturing other products, they did not have to waste energy on understanding the algorithm or on learning to choose between different algorithms. If they change jobs they learn a new pattern of working, and if the algorithm was replaced they had to learn a job description complying with the new rules. Not infrequently, our teaching materials employ exactly this kind of model. For example, we teach the pupils the algorithm for multiplication by decimals and we get

them to practise on tasks like 0.6×68 . After they have spent some time doing this, they come to a number of pages of "applied examples", such as "what is the price of 0.6 kg cheese at SEK 68/kg?" In these tasks, the pupils already know which algorithm they are to use: it is the one they have been practising on the previous pages.

Not infrequently, one even finds a hint in the chapter heading as to what the "applied tasks" will be concerned with applying. In principle, then, the pupils just have to carry on as before, which means looking to see which figures occur in the problem. This way the teacher is spared having to come to grips with their "everyday conceptions" of multiplication as leading to something which "comes to more".

Outside school life, however, nobody says that we are now going to do multiplication and now division. The individual needs a personal understanding of the relation between problem and mode of calculation. In addition, individuals in a modern society are often expected to be able to perform calculations unaided by standardised algorithms. In rough estimates with several different rules of arithmetic involved simultaneously, algorithms become unmanageable. In such contexts one needs a good perception of numbers and a good understanding of relations between the four rules of arithmetic. Instead of just being able to use and choose between several ready-made algorithms, one must also be able to construct new patterns of thought of one's own. This demands access to conceptual knowledge. Today we have hardly any time at all in school to concern ourselves with such knowledge, because the rules for manipulating figures and operational signs, i.e. algorithms, and the learning off by heart of the tables one has to know in order to handle them, pre-empt such a large proportion of teaching time.

The algorithms of the 21st century allow time to develop understanding

What happens when new, more functional algorithms crop up? Algorithms which people have not learned in school?

Many of us parents have been greatly confused by the new division algorithms that have succeeded one another in recent decades. These, however, can hardly be said to have given our children and teachers more

time to spend on the most important aspect of the subject of mathematics, namely conceptual knowledge.

New algorithms are being constructed all the time, both as regards the four rules of arithmetic and in other fields. Not all of them, however, seem as difficult and unnecessary as the two new division algorithms that have cropped up in our children's arithmetic books in recent decades in Sweden.

Unless the constant modernisation of algorithms is borne in mind, discussions of "algorithmisation" will be fruitless, not to say "old-fashioned". The time has now come, for example, to replace all the column algorithms we have learned with the extremely energy-efficient ones available today: the rules for using the pocket calculator. Most adults find the pocket calculator a positive experience. But what will be its impact on our children's mathematical conceptual development, an aspect which is already the Cinderella of the classroom? This is the question which anxious parents and teachers are asking.

Knowledge of tables and rules for working out are already built into the pocket calculator. Therefore the time taken to learn the new algorithms will be only a fraction of the time it took to learn the old ones. And so we will now have time to bridge the gap between arithmetical proficiency and "conceptual knowledge". We will have time to allow the pupils to integrate their school knowledge with the everyday knowledge they have already acquired out of school.

But do we know what to do with this time? What conceptual knowledge do the pupils need? And what everyday knowledge have they developed outside school before we even begin teaching them? What, in this everyday knowledge, do we need to help them to alter or deepen in order for their conceptual knowledge to be made functional? And how are we to do this? How have certain pupils managed to acquire advanced knowledge before they have even received any instruction? And how have they attached their equally well-developed arithmetical skills to this conceptual knowledge?

Only the pupils themselves can supply the answers to such questions.

I hope that the survey to be undertaken within the BUD project will reveal at least some of the answers.

Initial assumptions of the BUD project

My basic idea in planning the BUD project was to interview about 40 pupils at junior and intermediate levels in an attempt to chart untaught pupils' everyday conceptions of division and of fractions and decimals.

Another study was to include roughly the same number of pupils at intermediate and senior levels, the intention this time being to detect any misconceptions which, at worst, can result from the instruction received in school. A "map" of early conceptions and the ways in which they develop can, hopefully - together with a knowledge of the misconceptions our teaching is capable of leading to - be of assistance to teachers wishing to utilise the newfound teaching time for developing pupils' everyday conceptions further and gearing them to operations with the pocket calculator. But can a teacher keep track of all the conceptions which a map of this kind may conceivably describe?

Yes, many surveys of people's ways of understanding different phenomena have shown that there only exist a few conceptions of one and the same phenomenon - frequently no more than can be counted on the fingers of one hand. On the whole, for example, one finds only two conceptions of the earth in space when interviewing a group of pupils in compulsory school. The former geocentric view and the subsequent heliocentric (the former being still the more common of the two at senior level). The geocentric exhibits three or four different qualities: "we live inside the globe, which is split down the middle by the earth - that is, the ground", "we live on the outside of the globe, but only on the top of the earth," and so on.

The conceptions of different phenomena existing today among children and adults have been created and elaborated over centuries and millennia. Completely new conceptions are seldom created, and when they are the process takes a long time. Frequently it takes the intellectual exertions of several generations to create new conceptions. Those already existing, on the other hand, are re-fashioned and elaborated as time goes on. They

are re-created by children born into the communities where they exist. The children come up against the same problems as once obliged us to transform and further develop our understanding. But they do not need to spend as much time solving these problems as we once had to. After all, they are constantly encountering the much more advanced conceptions and theories already existing in the world around them. They come into contact with them when discussing their problems with adults, and they devote further thought to them when watching television, listening to the radio, reading books and so on. This is how they create and, gradually, transform their understanding of different phenomena.

Eventually they come to look on the surrounding world in roughly the same way as adults in their immediate environment, and perhaps they "upgrade" their conceptions to a level still closer to the latest scientific discoveries (3). When charting conceptions, it is the few qualitatively different ideas about a phenomenon existing on the path to this goal which one tries to expose and describe.

Relation of the new project to my own previous research

In my dissertation (4) I have charted the ways in which beginners' conceive of the first ten numbers, on the basis of these assumptions concerning the way in which conceptions are created and refashioned. The research approach underlying my assumptions, called phenomenography, was invented by researchers at the University of Göteborg. The 15% or so of beginners expressing the earliest conceptions looked on the number as an "extention". Numbers, for example, could quite simply mean "a lot" or "little".

In subsequent conceptions, the children had divided up this "extention" into units which they had made visible by making their fingers represent them. Two strong ideas are built into their use of fingers:

- (1) Don't split up the first hand (use the fingers as if they were the Roman numerals we once depicted them with). Then problems of the $2+ \_ = 9$ variety are solved by clenching the last two instead of the first two fingers, whereup the answer can be seen immediately, just as it would be if we were using Roman numerals to solve the same problem:

VII (II). The unknown part becomes the well-known finger symbol of "the whole hand plus two fingers.

- (2) The size of the parts of the number can quite well be changed by moving one or more fingers from one group of fingers to the other.

Many pupils had already begun "thinking with their hands" on these principles. And those who had made most progress knew, quite simply, that if one part of the number 9 was 2, the other must be 7, and that if one part was 3, the other must be 6, and so on.

The "map" which I derived from the interview study made a good foundation for teaching two of the interview classes in grades 1 and 2, because in the teaching experiment that was conducted we knew which problems to present in order for previous conceptions to come into the open, be discussed and gradually replaced with more adequate ones (3 and 4).

In a previous investigation, I had also taken an interest in the numerical conceptions of children who had been taught, in an attempt to catch sight of the changes undergone as a result of teaching. To this end I interviewed about 60 junior- and intermediate-level pupils, some 30 of whom were receiving special instruction in mathematics. The exploration of this group's way of apprehending numbers - and analysis of other research done in this field - revealed that about 15% of compulsory school pupils have difficulties with mathematics.

It looked as if the 15% or so of the pupils who, when starting school, only understood numbers after "3" in the sequence as expressions of "a lot" or "many" had never had an opportunity of developing these early conceptions any further.

Schools emphasise the development of counting skills, which has nothing whatsoever to do with the way in which these children perceive numbers. It is hardly possible to count out the answer to "little" + "much", and so the children appear, quite simply, to have learned rules for solving the sums in the book with the aid of the different types of concrete material occurring in school. In addition to this they had tried - and failed - to learn tables off by heart.

They had learned to "count" and were still doing so in grade 6. Their enumerations, in which they now began counting the words enumerated on their fingers, seemed, therefore, never likely to change into a thinking about numbers. The beginners who had themselves developed conceptions of numbers before starting school had used their fingers in a completely different way. They had "looked" at them instead of enumerating words, simultaneously counting these words with their fingers. The beginners' fingers had taken the place of the numerals and the children had fairly soon been able to imagine and, in their own bodies, feel the numbers and their parts until, finally, they simply knew how the numbers were related to each other, without any longer being aware of why they knew it. The more general conclusion I drew from this study was that it is important, when teaching begins, to know children's way of thinking about what we are trying to teach them. Only then can teaching in schools tie in with pupils' everyday conceptions and develop them further in a fruitful direction. And only if we know in what way teaching is liable to go wrong can we have any chance of avoiding methods leading to the development of "misunderstandings".

The further development of everyday conceptions or the creation of misconceptions of part of the whole

Most children in our society have already come across expressions like "half", "a quarter", and "a third" before starting school.

In a pilot study intended to form the basis of the main survey to be performed under the BUD project, I asked 15 pupils from grades 2, 3 and 4 to describe what they would like to give me "half", "a quarter" and "a third" of. I received almost as many suggestions as there were pupils. They included half an orange, a third of a banana and a quarter of a piece of string, half of "my marbles" (there were 100 of them), one-third of "my crayons" (64), a quarter of a twig, and so on.

The children were then asked to draw what they wanted to give me part of and to shade in the part they were giving me. It was difficult to shade in, say, a quarter of a twig or a third of a banana. They were then asked whether the parts had to be exactly equal in order for the shaded part to be rightly called "a third". Two of the pupils felt this was not

absolutely necessary for one-third but that it was necessary for a half or a quarter. All the others felt that the parts had to be exactly the same size. And all except one were able to think of something which they could give me half, a quarter or a third of. Hardly anybody, on the other hand, was able to describe a quarter and a third in figures. Only a couple of the pupils remembered seeing the expression " $1/2$ ".

Thus many of our pupils already know quite a lot about "part of the whole" before we start teaching them, but, having little idea of what this knowledge looks like, we are unable to make proper use of it.

Instead what often happens is that we start - through the textbook - by giving the pupils our own definitions and those pictures of the parts which we (or the author of the textbook) consider "good" and "illustrative".

The Dutch educationalist Hans Freudenthal (5), has sharply criticised the manner in which most textbooks introduce "part of the whole", namely by means of "pie slices". To give pupils an image, for example, of "one-third", textbooks usually show the picture of a round pie divided into three parts, whereas a square one divided into four is usually taken to be a good picture of "a quarter". Instead of giving the children pictures like this, intended to create number images, Freudenthal says that we ought to let them ponder for themselves what they could divide up, and how they would go about it. This way their conceptions of "part of the whole" could be made to link up with many different experiences, perhaps finally becoming abstract concepts of "thirds", "quarters" etc. From the interview replies described above, one senses that his ideas would go down excellently in the classroom.

Two Australian researchers (6) have shown, in an interview study of children in New Guinea, how "a third" and "a quarter" can be perceived through school instruction by children who have not developed any everyday knowledge of part of the whole outside school.

Numbers and parts of numbers do not occur in the everyday lives of children in New Guinea. "Pie-cutting" tuition was clearly reflected in the way these children solved problems. When, for example, they were

asked to give the interviewer a third or a quarter of a piece of string, they went to great pains trying to arrange the string into a circle divided into three parts in the first case, and into a square divided into four in the second case (Fig. 1).

Figure 1



This example helps us to gain an idea of the conceptions which it is possible for children with very early everyday conceptions or none at all to develop in our community. The textbooks used in New Guinea were no different from those used in the industrialised countries.

In the pilot study which I conducted myself, I also discovered how teaching can transform functional, everyday conceptions into "misunderstandings". Perhaps it was the textbook's way of defining the meaning of the figures in a fraction which made the instructed pupils "change down" to an inferior conception of the meaning behind the words which the figures are attached to, compared to the conception expressed by younger, untaught children.

Seven pupils in grade 6, interviewed at the same time as the 15 pupils from the junior grades, did not believe, for example, that $1/3$ of the figures in Fig. 2 were shaded, because these figures were divided into six parts.

Figure 2



In their view, two-sixths were shaded.

Asked whether this could also be called - or whether it could not also be - a third, they replied that "sixths can't be thirds". Nor did they understand how a fifth could be made into tenths, because "fifths can't turn into tenths". Of the ten pupils in grades 3 and 4 who were asked for their opinions on the same figures, only one did not spontaneously find that one-third was shaded in. For example, one of these untaught pupils explained, putting a finger over $2/6$ at a time, "you can see that there are three parts all the same size, and you've shaded one".

Principal aims of the BUD Project

When teaching begins, it ought logically to be an advantage to know something about the previous understanding possessed by pupils in the class and the misunderstandings which can develop as a result of our way of teaching. The aim of the survey undertaken within the BUD project is, if possible, to generate that kind of knowledge.

Briefly, I would like to describe the purpose of the "Children's conceptions of division and of equal parts", i.e. the BUD Project, as follows:

To chart early conceptions with development potential, advance and already viable conceptions and the transitional forms appearing in between, and to reveal misunderstandings liable to result from school instruction.

More exactly, my aim is, if possible, to chart:

1. How children start dividing before they know the multiplication table and the division algorithm.
2. How children begin to understand the connection between division and fractions, and the meaning of words and written expressions like "one quarter" - $1/4$. The explanation of these conceptions, of course, is also aimed at investigating children's conceptions of the way in which parts of numbers can be converted into different kinds of parts, and in the end how all kinds of fractions can be converted into tenths, hundreds etc., i.e. into decimals. Finally, an inquiry will be made into both older and younger pupils' conceptions of what happens if one operates with fractions and decimals (e.g. by multiplying and dividing them).
3. There is a third question to which I hope to obtain some kind of answer by means of a more long-term study in the form of a teaching experiment. This question concerns the way in which relations are created between children's conceptual knowledge of division and of rational numbers on the one hand and their "arithmetical skills" on the other. By arithmetical skills I mean their capacity for making intelligent use of the pocket calculator and for estimating - or mentally computing - the plausibility of the answers they get when

they use it. However, the aim of this teaching experiment should not be to illustrate the results obtained when the traditional column algorithms are replaced by the mini-calculator. The aim should be to illustrate how the time saved through the introduction of the mini-calculator could be used.

The better results obtained if the traditional algorithms are replaced by the mini-calculator have already been demonstrated in several Swedish studies, which show that the extensive time used for training algorithms gives no clear advantage. For example, in a large-scale investigation of pupils' arithmetic skills undertaken at the University of Uppsala in the early 1970s only 43% of the pupils in grade 9 could solve 84/175 correctly.

These projects, all parts of a larger investigation called the ARK project (Analys av Räknedosans Konsekvenser) have also illustrated that the mini-calculator gives time for teaching directed towards understanding (Björk and Brolin, 1984). A test intended to assess problem-solving skills was tried in randomly chosen classes at intermediate level, and in classes in which research into the use of the mini-calculator had been carried out. The tendency was, without exception, that the "research" classes achieved better results in the parts of the test that assessed conceptual knowledge as well as in the parts that tested arithmetic skills.

Given a better understanding of the informal ways in which children create arithmetical skills through increasing conceptual insights and how they increase their conceptual insights by exercising their newly acquired arithmetical skills, it is to be hoped that we will also be able, through the instruction given in schools, to achieve arithmetical skills which can be used outside school life: arithmetical skills relating to early everyday conceptions and to the more advanced conceptual knowledge into which these conceptions hopefully will be upgraded in school.

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